SUFFICIENT CONDITIONS FOR THE GENERAL PROBLEM OF MAYER WITH VARIABLE END POINTS*

BY

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1. **Introduction.** The problem of the calculus of variations to be considered here is the general problem of Mayer with variable end points as proposed by Bliss (V, p. 305)† and recently studied for a particular case in a joint paper by Bliss and Hestenes (XVI). As was remarked in the latter paper the general problem of Mayer is equivalent to the problem of Bolza, but the sets of sufficient conditions which have been given by Morse and Bliss for the problem of Bolza are not applicable to the problem of Mayer without further modification. In view of this fact it is the purpose of the present paper to establish a set of sufficient conditions for the general problem of Mayer with variable end points. The proofs here given are equally applicable to the problem of Bolza considered as a problem of Mayer.

The procedure used is similar to that used by Bliss for the problem of Bolza (XII, pp. 261–274). We first derive in §4 a further necessary condition analogous to that deduced by Bliss for the problem of Bolza. In §5 we construct an auxiliary problem of Mayer of the type discussed by Bliss and Hestenes (XVI). Their results are then applied in §§6 and 8 to the general problem of Mayer by methods closely related to those suggested by Mayer (XIII, pp. 436–465) and Hahn (XIV, pp. 127–136).

2. **Statement of the problem.** In the following pages the notation and the terminology used by Bliss and Hestenes for a particular problem of Mayer will be used throughout (XVI, pp. 306–309). In addition it will be understood that the indices μ, ν have the ranges

$$\mu, \nu = 1, \cdots, p < 2n + 1.$$

The general problem of Mayer is then that of minimizing a function $g[x_1, y(x_1), x_2, y(x_2)]$ in a class of arcs

$$y_i = y_i(x) \qquad (x_1 \leqq x \leqq x_2) \tag{2:1}$$

which satisfy the differential equations and end conditions

$$\phi_\alpha(x, y, y') = 0, \quad \psi_\mu[x_1, y(x_1), x_2, y(x_2)] = 0.$$

* Presented to the Society, April 8, 1932; received by the editors June 9, 1932, and, revised, December 10, 1932.

† The Roman numerals in the parentheses in the text refer to the bibliographies at the end of the paper by Bliss and Hestenes, cited here as XVI, and at the end of the present paper.

As before, the arcs (2:1) and the functions ϕ_α, g, ψ_μ will be assumed to have the continuity properties (a), (b), (c) (XVI, p. 306) in a neighborhood of a particular arc E_0 whose minimizing properties are to be studied, the determinant (2:1) appearing in (c) being now interpreted as a $(2n+2)\times(p+1)$-dimensional matrix of rank $p+1$.

For the general problem of Mayer the first necessary condition as given by Bliss and Hestenes (XVI, p. 307) is modified as follows, and is readily established by the methods which they suggest. The theorem has also been established by Morse and Myers (X, p. 245).

I. THE FIRST NECESSARY CONDITION. *Every minimizing arc E_0 for the problem of Mayer with variable end points must satisfy, besides the conditions* (XVI, p. 307)

$$F_{y_i'} = \int_{x_1}^{x} F_{y_i} dx + c_i, \quad \phi_\alpha = 0, \tag{2:2}$$

the further relation

$$(F - y_i' F_{y_i'})dx + F_{y_i'} dy_i \Big|_1^2 + \lambda_0 dg = 0 \tag{2:3}$$

for every set of differentials dx_1, dy_{i1}, dx_2, dy_{i2} satisfying the equations $d\psi_\mu = 0$, λ_0 being a suitably chosen constant.

An admissible arc E_0 is said to be *normal relative to the end conditions* $\psi_\mu = 0$ if there exist for it p sets of admissible variations ξ_1^ν, ξ_2^ν, $\eta_i^\nu(x)$ such that the determinant $|\Psi_\mu(\xi^\nu, \eta^\nu)|$ is different from zero (XVI, p. 307). For convenience an arc that is normal relative to the end conditions $\psi_\mu = 0$ will be designated simply as *normal.*

THEOREM 2:1. *An admissible arc that does not satisfy the necessary condition* I *is normal.*

This follows at once because an admissible arc E_0 satisfies the necessary condition I if and only if every determinant of the form

$$\begin{vmatrix} G(\xi^\sigma, \eta^\sigma) \\ \Psi_\mu(\xi^\sigma, \eta^\sigma) \end{vmatrix} \qquad (\sigma = 1, \cdots, p+1)$$

vanishes, where ξ_1^σ, ξ_2^σ, $\eta_i^\sigma(x)$ are $p+1$ sets of admissible variations for E_0, and the function G is obtained from g in the same manner as Ψ_μ is obtained from ψ_μ (V, p. 309).

THEOREM 2:2. *An admissible arc E_0 that satisfies the necessary condition* I *is normal if and only if there exist for it no set of multipliers $\lambda_\alpha(x)$, not vanishing* *simultaneously, with which it satisfies equations* (2:2) *and for which all* $(p+1)$-*rowed determinants of the matrix*

$$\left\| \begin{matrix} -y'_{i1}F_{y_i'}(x_1) & F_{y_i'}(x_1) & y'_{i2}F_{y_i'}(x_2) & -F_{y_i'}(x_2) \\ \psi_{\mu x_1} & \psi_{\mu y_{i1}} & \psi_{\mu x_2} & \psi_{\mu y_{i2}} \end{matrix} \right\| \tag{2:4}$$

vanish. If E_0 *is normal the constant* λ_0 *can be chosen to be unity, the multipliers* $\lambda_\alpha(x)$ *with which* E_0 *satisfies the conditions* (2:2) *and* (2:3) *being then unique.*

This theorem is an obvious generalization of a theorem given by Bliss and Hestenes and can be proved by the same methods (XVI, p. 308). A similar theorem has been established by Bolza (III, p. 441).

3. **Theorems on extremals.** It is known that in the problems of Mayer a non-singular extremal arc can be imbedded in a $(2n-1)$-parameter family of extremals (XVI, p. 311)

$$y_i = y_i(x, c_1, \cdots, c_{2n-1}),\ \lambda_\alpha = \lambda_\alpha(x, c_1, \cdots, c_{2n-1}) \qquad (x_1 \leqq x \leqq x_2). \tag{3:1}$$

Further properties of this family are given in the following theorem:

THEOREM 3:1. *Let* E_0 *be a member of the* $(2n-1)$-*parameter family of extremals* (3:1) *for parameter values* (x_{10}, x_{20}, c_0). *If the matrix*

$$\left\| \begin{matrix} y_{ic_s}(x_1, c) \\ y_{ic_s}(x_2, c) \end{matrix} \right\| \tag{3:2}$$

has rank $2n-1$ *on* E_0, *then there is a neighborhood* N *of the ends of* E_0 *in* $(x_1y_1x_2y_2)$-*space such that the end values of every extremal of the family* (3:1) *with ends in* N *satisfy a relation* $W(x_1, y_1, x_2, y_2) = 0$. *Conversely, every pair of points* (x_1, y_1), (x_2, y_2) *in* N *satisfying the condition* $W=0$ *can be joined by an extremal* E *of the family* (3:1), *and by taking* N *sufficiently small the parameters* (x_1, x_2, c) *belonging to* E *will lie in a preassigned* ϵ-*neighborhood of those belonging to* E_0. *The function* W *has continuous partial derivatives of the first two orders in* N.

The theorem can be proved as follows. Select $2n$ constants a_i, b_i such that the determinant

$$\left| \begin{matrix} y_{ic_s}(x_1, c) & a_i \\ y_{ic_s}(x_2, c) & b_i \end{matrix} \right| \tag{3:3}$$

is different from zero on E_0. Consider now the equations

$$\begin{aligned} y_{i1} &= y_i(x_1, c) + Wa_i, \\ y_{i2} &= y_i(x_2, c) + Wb_i. \end{aligned} \tag{3:4}$$

These equations are satisfied by the set $(x_{10}, y_{10}, x_{20}, y_{20}, c_0, W=0)$ belonging to E_0. Furthermore the functional determinant with respect to the variables c_s, W is the determinant (3:3) and is therefore different from zero on E_0. It follows that equations (3:4) have a unique solution

$$c_s = c_s(x_1, y_1, x_2, y_2), \quad W = W(x_1, y_1, x_2, y_2) \tag{3:5}$$

in a neighborhood N of the end values $(x_{10}, y_{10}, x_{20}, y_{20})$ belonging to E_0. The right members of equations (3:5) have continuous first and second derivatives in N since the right and left members of equations (3:4) have such derivatives. If now the end values of an extremal are in N, then these end values must satisfy the relation $W(x_1, y_1, x_2, y_2)=0$ since the solutions of equations (3:4) are unique. Furthermore every set of values (x_1, y_1, x_2, y_2) in N satisfying the relation $W=0$ are the end values of an extremal E with parameter values $[x_1, x_2, c(x_1, y_1, x_2, y_2)]$, and by taking N sufficiently small these parameter values will lie in a preassigned ϵ-neighborhood of those belonging to E_0. Hence the theorem is proved.

It is now possible to give an interesting geometric interpretation of normality.

THEOREM 3:2. *A non-singular extremal arc E_0 whose matrix* (3:2) *has rank $2n-1$ is normal if and only if in the space of points (x_1, y_1, x_2, y_2) the extremal end manifold $W=0$ and the terminal manifold $\psi_\mu=0$ are not tangent to each other at the point $(x_{10}, y_{10}, x_{20}, y_{20})$ defining the end values of E_0.*

To prove this it is sufficient, as is readily seen, to show that the derivatives $W_{x_1}, W_{y_{i1}}, W_{x_2}, W_{y_{i2}}$ are proportional to the elements of the first row of the matrix (2:4). These derivatives have this property because the relation $F_{y_i'}\eta_i$ $=$constant along extremals (XVI, p. 307) with $\eta_i = y_{ic_s} dc_s$ implies that the differentials $dx_1, dy_{i1}, dx_2, dy_{i2}, dc_s, dW$ belonging to equations (3:4) satisfy the relation

$$F_{y_i'}(dy_i - y_i' dx)\Big|_1^2 = F_{y_i'} y_{ic_s} dc_s \Big|_1^2 + h\,dW = h\,dW$$

where $h = b_i F_{y_i'}(x_2) - a_i F_{y_i'}(x_1)$. If $h=0$ on E_0 then on account of the relation $F_{y_i'}\eta_i=$constant, the determinant (3:3) would vanish on E_0 which is not the case. Hence $h \neq 0$ on E_0 and

$$\begin{aligned} y_{i1}' F_{y_i'}(x_1) &= hW_{x_1}, & -F_{y_i'}(x_1) &= hW_{y_{i1}}, \\ -y_{i2}' F_{y_i'}(x_2) &= hW_{x_2}, & F_{y_i'}(x_2) &= hW_{y_{i2}} \end{aligned} \tag{3:6}$$

as was to be proved.

4. **The necessary condition of Mayer.** The necessary condition of Mayer for the problem of Bolza, as stated by Bliss (XII, p. 266), is also valid for the problem of Mayer considered here.* In order to derive this condition we sup-

* The proof is somewhat different from that given by Bliss for the problem of Bolza. He has called my attention to the fact that the argument which he used is inadequate in the case when the ends of E_0 are conjugate, and has suggested the modifications indicated here.

pose that E_0 is a normal non-singular minimizing arc without corners and hence an extremal arc. If $\xi_1, \xi_2, \eta_i(x)$ are a set of admissible variations for E_0 which satisfy the conditions $\Psi_\mu(\xi, \eta)=0$, then E_0 is a member of a one-parameter family of admissible arcs with ends satisfying the conditions $\psi_\mu=0$ and having $\xi_1, \xi_2, \eta_i(x)$ as its variations along E_0 (IX, p. 695). For such a family the second variation of the function g to be minimized is expressible along E_0 in the form

$$(4:1) \qquad I_2 = (F_x + y_i' F_{y_i})\xi^2 + 2F_{y_i}\eta_i\xi \Big|_1^2 + 2(Q + l_\mu Q_\mu) + \int_{x_1}^{x_2} 2\omega(x, \eta, \eta')dx$$

where Q, Q_μ are quadratic forms in $\xi_1, \eta_i(x_1), \xi_2, \eta_i(x_2)$ whose coefficients are the second derivatives of the functions g, ψ_μ, respectively, and

$$2\omega(x, \eta, \eta') = F_{y_i y_k}\eta_i\eta_k + 2F_{y_i y_k'}\eta_i\eta_k' + F_{y_i' y_k'}\eta_i'\eta_k' .$$

This form for I_2 is readily obtained with the help of the transversality condition (2:3) by the methods used by Bliss and Hestenes (XVI, pp. 311–312). Let us consider variations satisfying the equations $\Psi_\mu(\xi, \eta)=0$ along E_0, and of the special form $\xi_1=dx_1$, $\xi_2=dx_2$, $\eta_i=\delta y_i=y_{ic_s}dc_s$, where the functions $y_i(x, c)$ are those defining the $(2n-1)$-parameter family (3:1) of extremals to which E_0 belongs. For such variations the second variation (4:1) can also be expressed in the form

$$(4:2) \qquad \begin{aligned} d^2g &= (F_x + y_i' F_{y_i})dx^2 + 2F_{y_i}\delta y_i dx \\ &\quad + \delta y_i \Omega_{\eta_i'}(x, \delta y, \delta y', \delta\lambda)\Big|_1^2 + 2(Q + l_\mu Q_\mu) \end{aligned}$$

given by Bliss (XII, p. 266), where $\delta\lambda_\alpha=\lambda_{\alpha c_s}dc_s$ and

$$\Omega(x, \eta, \eta', \mu) = \omega(x, \eta, \eta') + \mu_\alpha(\phi_{\alpha y_i}\eta_i + \phi_{\alpha y_i'}\eta_i').$$

Since E_0 is a minimizing arc the expression (4:1) must be $\geqq 0$ for all sets of admissible variations $\xi_1, \xi_2, \eta_i(x)$ which satisfy the conditions $\Psi_\mu(\xi, \eta)=0$. In particular it must be $\geqq 0$ for variations $\xi_1=dx_1$, $\xi_2=dx_2$, $\eta_i=\delta y_i$ of the special type considered above satisfying the conditions $d\psi_\mu \equiv \Psi_\mu(dx, \delta y)=0$. We have therefore the following result:

IV. THE NECESSARY CONDITION OF MAYER. *For a normal non-singular minimizing arc E_0 without corners the quadratic form* (4:2) *must satisfy the condition $d^2g \geqq 0$ for all sets $(dx_1, dx_2, dc_s) \neq (0, 0, 0)$ which satisfy the equations $d\psi_\mu=0$. Furthermore between the end points* 1 *and* 2 *on E_0 there can be no point* 3 *conjugate to* 1 *defined by a value x_3 such that E_0 is normal on the interval x_3x_2.*

The last statement is a slight modification of the condition IV deduced by Bliss and Hestenes for problems of Mayer having $2n+1$ end conditions (XVI, p. 315), valid here for E_0 since E_0 must also be a minimizing arc for such a problem, as will be seen in the next section.

5. **An auxiliary problem of Mayer.** In order to construct a problem of Mayer of the type described in the last paragraph we suppose that E_0 is a minimizing arc for the general problem of Mayer considered here. Its end values $(x_{10}, y_{10}, x_{20}, y_{20})$ satisfy the conditions $\psi_\mu = 0$ $(\mu = 1, \cdots, p)$. Adjoin to the functions ψ_μ, $2n+1-p$ functions $\psi_\tau(x_1, y_1, x_2, y_2)$ $(\tau = p+1, \cdots, 2n+1)$ possessing continuous first and second partial derivatives in a neighborhood of the values $(x_{10}, y_{10}, x_{20}, y_{20})$, vanishing at these values, and having the determinant

$$\begin{vmatrix} g_{x_1} & g_{y_{i1}} & g_{x_2} & g_{y_{i2}} \\ \psi_{\rho x_1} & \psi_{\rho y_{i1}} & \psi_{\rho x_2} & \psi_{\rho y_{i2}} \end{vmatrix} \tag{5:1}$$

different from zero on E_0. The new set of end conditions $\psi_\rho = 0$ $(\rho = 1, \cdots, 2n+1)$ defines an auxiliary problem of Mayer of the type discussed by Bliss and Hestenes. It is clear that E_0 is also a minimizing arc for this auxiliary problem.

THEOREM 5:1. *Let E_0 be an admissible arc that is normal on the interval $x_{10}x_{20}$ and satisfies the necessary condition* I. *If E_0 is normal relative to the end conditions $\psi_\mu = 0$ $(\mu = 1, \cdots, p)$, then it is normal relative to the end conditions $\psi_\rho = 0$ $(\rho = 1, \cdots, 2n+1)$ just defined.*

To prove this theorem we recall that the matrix (2:4) has rank $p+1$ since E_0 is normal relative to the end conditions $\psi_\mu = 0$. Furthermore since E_0 satisfies the transversality condition (2:3), it follows that on E_0 the derivatives $g_{x_1}, g_{y_{i1}}, g_{x_2}, g_{y_{i2}}$ are expressible as a linear combination of the rows of the matrix (2:4), the multiplier of the first row being different from zero. The rank of the matrix (2:4) formed for the new end conditions $\psi_\rho = 0$ is therefore unaltered when the elements of the first row are replaced by the derivatives $g_{x_1}, g_{y_{i1}}, g_{x_2}, g_{y_{i2}}$. The matrix thus formed is the matrix of the determinant (5:1) and has rank $2n+2$. Hence according to Theorem 2:2, E_0 is also normal relative to the end conditions $\psi_\rho = 0$, and the theorem is established.

6. **A fundamental sufficiency theorem.** With the help of the auxiliary problem just constructed we can prove the following theorem:

THEOREM 6:1. A FUNDAMENTAL SUFFICIENCY THEOREM. *Let E_0 be an extremal arc with the following properties:*

(A). *E_0 satisfies the sufficient conditions for a proper strong relative minimum* *with respect to admissible arcs C satisfying the end conditions $\psi_\rho(C)=0$ of the auxiliary problem of Mayer defined in* §5.

(B) *There is a neighborhood M of the ends of E_0 in $(x_1y_1x_2y_2)$-space such that the inequality $g(E)>g(E_0)$ holds for every extremal E of the family* (3:1) *with ends in M satisfying the conditions $\psi_\mu(E)=0$ and not identical with E_0.*

Then there exist neighborhoods $\mathfrak{F}$ of E_0 in xy-space and N of the ends of E_0 in $(x_1y_1x_2y_2)$-space such that the inequality $g(C)>g(E_0)$ holds for every admissible arc C in $\mathfrak{F}$ with ends in N satisfying the conditions $\psi_\mu(C)=0$ and not identical with E_0.

The proof is based on the following two lemmas, the proofs of which will be given in the next section.

LEMMA 6:1. (Modification of Hahn's Theorem (XIV, p. 129).) *The property* (A) *for E_0 implies the existence of neighborhoods $\mathfrak{F}$ of E_0 in xy-space and M of the ends of E_0 in $(x_1y_1x_2y_2)$-space such that for every extremal E of the family* (3:1) *with ends in M the inequality $g(C)>g(E)$ holds for every admissible arc C in $\mathfrak{F}$ with ends in M satisfying the conditions $\psi_\rho(C)=\psi_\rho(E)$ and not identical with E.*

LEMMA 6:2. *The property* (A) *for E_0 implies that every neighborhood M of the end values of E_0 has associated with it a second neighborhood N of these end values such that for every admissible arc C with ends in N there is an extremal E of the family* (3:1) *with ends in M satisfying the conditions $\psi_\rho(C)=\psi_\rho(E)$.*

With the help of these lemmas the proof of Theorem 6:1 is as follows. Select first neighborhoods $\mathfrak{F}$ of E_0 and M of the ends of E_0 effective as in Lemma 6:1 and as in (B). Select a second neighborhood N of the ends of E_0 related to M as in Lemma 6:2. Consider now an admissible arc C in $\mathfrak{F}$ with ends in N satisfying the conditions $\psi_\mu=0$. According to Lemma 6:2 there is an extremal E of the family (3:1) with ends in M satisfying the conditions $\psi_\mu(E)=0$, $\psi_\tau(E)=\psi_\tau(C)$, where the functions ψ_τ are those adjoined to the functions ψ_μ to form the auxiliary Mayer problem defined in §5. From Lemma 6:1 it follows that $g(C)\geqq g(E)$, and from the property (B) we have $g(E)\geqq g(E_0)$. Hence $g(C)\geqq g(E_0)$, the equality being valid only in case C coincides with E_0, as was to be proved.

7. **Proofs of two lemmas.** In order to prove Lemma 6:1 we use the result obtained by Bliss and Hestenes (XVI, p. 323)* which states that the

* In the proof of the theorem referred to here, the authors made use (XVI, Theorem 8:1) of a suggestion in an abstract by Morse, Bulletin of the American Mathematical Society, vol. 37 (1931), p. 37. Bliss and Reid proved Morse's result independently before the complete paper of Morse (XVII) appeared. Bliss and Hestenes used the proof given by Bliss, which is similar to that of Morse, and inadvertently made no reference to Morse's paper. The proof given by Morse should of course have priority.

property (A) for E_0 given in Theorem 6:1 implies the existence of a function $W(a_1, \cdots, a_n)$ such that the n-parameter family of extremals

$$(7:1) \qquad y_i = y_i(x, x_{20}, a, W_a), \quad z_i = z_i(x, x_{20}, a, W_a) \qquad (x_1 \leqq x \leqq x_2)$$

contains E_0 for parameter values (x_{10}, x_{20}, a_0) and has the determinant $|y_{ia_k}|$ different from zero along E_0. Furthermore each extremal E of the family (7:1) has on it the element $(x, y, z) = (x_{20}, a, W_a)$, where the a_i are the parameter values defining E. If now we select $n-1$ functions $W_r(a_1, \cdots, a_n)$ having continuous first and second partial derivatives and such that the determinant $|W_{a_i} W_{ra_i}|$ is different from zero for the values $a_i = a_{i0}$, then the $(2n-1)$-parameter family of extremals

$$(7:2) \qquad \begin{aligned} y_i &= y_i(x, x_{20}, a, W_a + b_r W_{ra}) = y_i(x, a, b), \\ z_i &= z_i(x, x_{20}, a, W_a + b_r W_{ra}) = z_i(x, a, b) \qquad (x_1 \leqq x \leqq x_2) \end{aligned}$$

contains E_0 for parameter values $(x_{10}, x_{20}, a_0, b=0)$. Moreover every extremal E of this family has on it the element $(x, y_i, z_i) = (x_{20}, a_i, W_{a_i} + b_r W_{ra_i})$, where the parameter values a_r, b_r are those defining E. The equations expressing this fact are the equations

$$a_i = y_i(x_{20}, a, b), \quad W_{a_i} + b_r W_{ra_i} = z_i(x_{20}, a, b),$$

and by differentiation it is found that the determinant

$$\begin{vmatrix} y_{ia_k} & y_{ib_r} & 0 \\ z_{ia_k} & z_{ib_r} & z_i \end{vmatrix}$$

is different from zero for the values $(x, a, b) = (x_{20}, a_0, 0)$. Hence the family (7:2) is one of the type (3:1), its multipliers $\lambda_\alpha(x, a, b)$ being found in the usual manner (XVI, pp. 309–311).

Since the determinant $|y_{ia_k}|$ belonging to the family (7:1) is different from zero on E_0, the determinant $|y_{ia_k}(x, a, b)|$ belonging to the family (7:2) has the same property. Hence the system of equations

$$(7:3) \qquad y_i = y_i(x, a, b)$$

has a unique solution

$$a_i = a_i(x, y, b)$$

in a neighborhood $\mathfrak{D}$ of the values (x, y, b) belonging to E_0. The functions $a_i(x, y, b)$ are continuous and possess continuous derivatives of the first two orders in the domain $\mathfrak{D}$. If now we let

$$(7:4) \qquad \begin{aligned} p_i(x, y, b) &= y_{ix}[x, a(x, y, b), b], \\ \lambda_\alpha(x, y, b) &= \lambda_\alpha[x, a(x, y, b), b], \end{aligned}$$

then according to the condition $\text{II}_{\mathfrak{N}}'$ implied by the property (A) on E_0, the domain $\mathfrak{D}$ can be so restricted that at each element (x, y, b) in $\mathfrak{D}$ the inequality

$$E[x, y, p(x, y, b), \lambda(x, y, b), y'] > 0$$

holds for every admissible set $(x, y, y') \neq (x, y, p)$, where $E(x, y, p, \lambda, y')$ is the Weierstrass E-function (XVI, pp. 317, 324). Furthermore on the hyperplane $x = x_{20}$ in xy-space the Hilbert integral I^* is independent of the path when the parameters b_r are fixed (XVI, p. 323, cf. XII, p. 269). It follows that for each set b_r the region $\mathfrak{F}$ of points (x, y), whose elements (x, y, b) are all in $\mathfrak{D}$, forms a field with slope functions and multipliers defined by equations (7:4) (XVI, p. 322). We have a family of such fields depending upon the $n-1$ parameters b_r. In each field the Weierstrass E-function is >0 unless $y_i' = p_i$. Hence according to a theorem proved by Bliss and Hestenes (XVI, p. 319) there is a neighborhood M of the end values of E_0 such that every extremal E with ends in M and belonging to one of these fields furnishes a proper strong relative minimum for the function g in the class of admissible arcs C in $\mathfrak{F}$ whose ends are in M and satisfy the conditions $\psi_\rho(C) = \psi_\rho(E)$.

Lemma 6:1 will now be established completely if we show that the neighborhood M of the ends of E_0 can be restricted so that every extremal E of the family (7:2) with ends in M is a member of one of the fields just described. To do this we select a constant h so that the set $[x, y, a_i(x, y, b), b]$ with elements (x, y, b) in $\mathfrak{D}$ is the only solution of equations (7:3) satisfying the relation

$$a_i(x, y, b) - h \leqq a_i \leqq a_i(x, y, b) + h. \tag{7:5}$$

This can always be done since the solution $a_i(x, y, b)$ of equations (7:3) is isolated. We now select a constant ϵ such that the inequalities

$$| a_i - a_{i0} | < h/2,$$
$$| a_{i0} - a_i(x, y, b) | < h/2$$

hold along every extremal E of the family (7:2) with parameter values (x_1, x_2, a, b) in an ϵ-neighborhood of those belonging to E_0. The relation (7:5) now holds for every set of values (x, y, a, b) on E. It follows that $a_i = a_i(x, y, b)$, and hence E is an extremal of one of the fields just described. This completes the proof of Lemma 6:1 since according to Theorem 3:1 the neighborhood M of the ends of E_0 can be so restricted that every extremal E of the family (7:2) with ends in M has parameter values (x_1, x_2, a, b) in the ϵ-neighborhood just defined.

In order to prove Lemma 6:2 consider first the equations

$$W(x_1, y_1, x_2, y_2) = 0, \qquad (7:6)$$
$$\psi_\rho(x_1, y_1, x_2, y_2) = m_\rho,$$

where W is the function defined in Theorem 3:1. As was seen in §3, the functional determinant of these equations is different from zero on E_0. Furthermore equations (7:6) are satisfied by the set $(x_1, y_1, x_2, y_2, m) = (x_{10}, y_{10}, x_{20}, y_{20}, 0)$ belonging to E_0. Hence there is a constant $h>0$ such that equations (7:6) have a unique solution

$$x_1 = x_1(m), \quad y_{i1} = y_{i1}(m), \qquad (7:7)$$
$$x_2 = x_2(m), \quad y_{i2} = y_{i2}(m)$$

for all values m_ρ satisfying the relations $|m_\rho| < h$. If h is sufficiently small, then according to Theorem 3:1 every pair of points (x_1, y_1), (x_2, y_2) can be joined by an extremal of the family (3:1). Furthermore it is clear that, if necessary, the constant h can be further restricted so that every set of values (x_1, y_1, x_2, y_2) defined by equations (7:7) with $|m_\rho| < h$ is in a preassigned neighborhood M of the end values of E_0. If now we select a second neighborhood N of the end values of E_0 so that every set of values (x_1, y_1, x_2, y_2) in N satisfies the relation $|\psi_\rho(x_1, y_1, x_2, y_2)| < h$, then every admissible arc C with ends in N determines a set of values $m_\rho = \psi_\rho(C)$ satisfying the relation $|m_\rho| < h$, and these in turn determine an extremal arc E with ends in M satisfying the conditions $\psi_\rho(E) = \psi_\rho(C)$. This proves Lemma 6:2.

8. **Sufficient conditions for relative minima.** The necessary condition I is given in §2. The symbols $\text{II}_{\mathfrak{N}}'$, III' will be used to denote the strengthened conditions of Weierstrass and Clebsch as defined by Bliss and Hestenes (XVI, p. 324). The symbol IV' will be used to denote the condition IV of §4 strengthened so as to exclude the equality sign. With these definitions agreed upon we can state the following theorem:

THEOREM 8:1. SUFFICIENT CONDITIONS FOR A STRONG RELATIVE MINIMUM. *Let E_0 be an admissible arc without corners and with end points determined by values x_{10}, x_{20} and satisfying the conditions $\psi_\mu = 0$. If E_0 is normal relative to the end conditions $\psi_\mu = 0$, is normal on every sub-interval $x_{10}x_3$ of $x_{10}x_{20}$, and satisfies the conditions* I, $\text{II}_{\mathfrak{N}}'$, III', IV', *then there exist neighborhoods $\mathfrak{F}$ of E_0 in xy-space and N of the ends of E_0 in $(x_1y_1x_2y_2)$-space such that the inequality $g(C) > g(E_0)$ holds for every admissible arc C in $\mathfrak{F}$ with ends in N satisfying the conditions $\psi_\mu(C) = 0$ and not identical with E_0.*

The theorem will be established if we can show that the hypotheses of the theorem imply those of Theorem 6:1. It is easily seen from Theorem 5:1 and from the sufficiency conditions given by Bliss and Hestenes for the case $p=2n+1$ (XVI, p. 324) that E_0 is an extremal arc having the property (A) of Theorem 6:1 provided that we can show that the condition IV', as defined above, implies that the ends of E_0 are not conjugate to each other. If the ends of E_0 were conjugate then the constants dc_s in the expressions $\delta y_i = y_{ic_s} dc_s$ could be selected not all zero so that the differentials δy_i would all vanish at the ends of E_0. If we should take these constants dc_s together with the values $dx_1 = dx_2 = 0$, then the conditions $d\psi_\mu = 0$ would be satisfied and the expression (4:2) for $d^2 g$ would vanish, which would contradict the condition IV'. Hence E_0 has property (A) of Theorem 6:1.

To prove that E_0 has the property (B) of Theorem 6:1 we first note that the conditions I, III' imply the existence of a family of extremals (3:1) containing E_0 for parameter values (x_{10}, x_{20}, c_{s0}). From conditions I, IV' it follows that $dg=0$, $d^2g>0$ for every set of differentials $(dx_1, dx_2, dc_s) \neq (0, 0, 0)$ which satisfy the conditions $d\psi_\mu = 0$. But these are the conditions (XV, p. 115) which insure that $g(x_1, x_2, c_s) > g(x_{10}, x_{20}, c_{s0})$ for all sets $(x_1, x_2, c_s) \neq (x_{10}, x_{20}, c_{s0})$ satisfying the equations $\psi_\mu(x_1, x_2, c_s) = 0$ and lying in a sufficiently small ϵ-neighborhood of (x_{10}, x_{20}, c_{s0}). Furthermore since the ends of E_0 are not conjugate the matrix (3:2) has rank $2n-1$ (XVI, p. 316), and according to Theorem 3:1 there is a neighborhood M of the ends of E_0 such that every extremal with ends in M has parameter values (x_1, x_2, c_s) in the ϵ-neighborhood described above. It follows that $g(x_1, y_1, x_2, y_2) > g(x_{10}, y_{10}, x_{20}, y_{20})$ for every extremal E with ends in M satisfying the conditions $\psi_\mu(E) = 0$ and not identical with E_0. Hence E_0 has the property (B) of Theorem 6:1 and Theorem 8:1 is established.

In a similar manner sufficient conditions for a weak relative minimum for the general problem of Mayer with variable end points can be established. The argument is like that of Bliss and Hestenes (XVI, p. 325) with the help of simple modifications of Theorem 6:1 and Lemma 6:1 above. The Theorem 10:2 of Bliss and Hestenes remains valid here if we replace the phrase "preceding theorem" by "Theorem 8:1" and the equations $\psi_\rho = 0$ by $\psi_\mu = 0$. Similarly Corollary 10:1 of the paper by Bliss and Hestenes is still effective if we replace "Theorem 10:1" by "Theorem 8:1" and ψ_ρ by ψ_μ.

Bibliography

The papers listed below are a continuation of the list at the end of the paper of Bliss and Hestenes cited here as XVI.

XIII. Mayer, *Zur Aufstellung der Kriterion des Maximums und Minimums der einfachen Integrale bei variablen Grenzwerten*, Leipziger Berichte, vol. 36 (1884), pp. 99–128, vol. 48 (1896), pp. 436–465.

XIV. Hahn, *Ueber Variationsprobleme mit variablen Endpunkten*, Monatshefte für Mathematik und Physik, vol. 22 (1911), pp. 127–136.

XV. Hancock, *Theory of Maxima and Minima*, Ginn and Company, 1917.

XVI. Bliss and Hestenes, *Sufficient conditions for a problem of Mayer in the calculus of variations*, these Transactions, vol. 35 (1933), pp. 305–326.

XVII. Morse, *Sufficient conditions in the problem of Lagrange with fixed end points*, Annals of Mathematics, (2), vol. 32 (1931), pp. 567–577.

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